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# Analytic Method for Vibration Analysis of Track Structure Induced by High-Speed Train

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## Abstract

Nowadays, the increase in heavy freight rail transport and high-speed train (HST) operations has encouraged scientists to investigate the dynamic response of rail structures under moving load using analytically and numerically computational methods. In this study, to analyse vibration of rail structure, the rail has been modelled as continuous Euler-Bernoulli elastic beam system. The effects of some basic parameters such as track foundation elasticity modulus, rail stiffness, wheel set axle load and rail critical velocity which affect rail vibrations, on vibrations were examined in detail by considering different track foundation properties. The vibration wave amplitude increases as the train speed approaches the rail infrastructure critical speed.

**Keywords:** HST, moving load, elastic foundation, continuous beam

## 1. INTRODUCTION

The increasing interest in high-speed rail transport in the last decade has led to more efforts to investigate the dynamic interaction between rail vehicle and rail structure. Studies on railway dynamics generally consist of models created to study rail vibrations. These rail models have been divided into two main categories in the literature: continuously supported rail beams and discrete supported rail beams [1]. In these models, rail, rail pads, fasteners, sleepers, ballast and ground are the components that define the value of the rail modulus.

The rail on the elastic foundation has been modelled according to Euler-Bernoulli beam theory in recent studies [2]. Euler-Bernoulli and Timoshenko beam theories are two most common used beam models in literature studies to research dynamic response of track effect of HST [2–5].

To investigate rail dynamic response, the first model is studied by Winkler in 1867. This model consist of a rail model which placed on the elastic foundation [6]. The elastic foundation of the rail track has been modelled as uniformly distributed linear spring to represent all rail components.

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Duffy [7] presented transient and steady state solutions of the Euler-Bernoulli beam with elastic foundation taking into moving mass inertial effect. Grassie and Cox [8] showed that the large stresses generated in the rail beam, which they analysed as a continuous support beam model, are related to low damping sleeper resonance. A study investigating effect of the rail pad stiffness upon sleeper stresses has been given by [9]. Consequently, it was proven that soft rail pad more isolate to vibration than hard pad and sleeper stresses measured soft rail pad is smaller than hard pad. Modern railway tracks are characterized by correctly placed and continuously welded rails and sleepers. In this technique, mechanical modelling of the railway track has been made very simple by considering railway track as two straight parallel beams supported by sleepers[10].Continuous supported rail models contains many dynamic behaviour parameter in order to have all properties of the railway tracks[11]. The discrete rail models are more appropriate because of conventional railway rails are discretely supported by rail pads and sleepers. Grassie *et al.* [12] investigated to railway track considering two different models with one continuous and the other incorporating the mass of sleepers in the frequency range from 50 to 1500 Hz. Hunt [13] modelled to rails on the railway as a pair of Euler-Bernoulli beams placed side by side on an elastic foundation.

Nowadays, it is very common technique to use computer programs in order to model, design and analyse any engineering structure. The computers present big advantages for scientist so as to model complex engineering problems and analyse due to having big capacity and presenting high performance processing. In previous studies [14,15] only one moving axle load is considered on the elastic structure for HST analysis. In this study, the mathematically model and computer simulation of the rail beam which modelled according to elastic beam theory and moving HST passing over the rail beam have been presented. The effect of some parameters such as rail stiffness, track ballast elasticity module, train wheelset load and HST's critical velocity upon rail displacement has been investigated in detail. In the context of this article, the mathematical formulation for physical model of the rail beam on the elastic foundation has been introduced in section 2. At the same time, in section 3, the effect of some parameters such as rail, elastic foundation and HST upon rail displacement has been investigated using some numerical examples considering different HST velocity and track foundation properties.

## 2. MATHEMATICAL FORMULATION

In this study, the physical model shown in Figures1a-b have been introduced to examine rail vibration effect of moving HST on the elastic track foundation. Figure 1a shows continuous elastic beam models for unloaded railway track superstructure. Similarly, Figure 1b presents rail beam effect of point moving load *P* with elastic foundation.

### 2.1. Analytic Formulation of Track Structure Considering Elastic Beam Model

The time dependent equation of motion for undamped continuous elastic rail beam given by Figure 1b has been derived as shown in Equation (1) using Hamilton principle considering moving on its single wheelset point load *F* [14,15].

$$EI\frac{\partial^4 w}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}+kw=-F\delta\left(x-vt\right) \qquad (1)$$

The parameters $E$ and $I$ in Equation (1) represent elasticity module of rail beam and inertial moment of the beam cross-section on the horizontal axis respectively. In addition to this, the parameter $w$ represents displacement of the rail beam at any point $x$ on location beam and at any time $t$. The parameters $m$, $k$ and $\delta$ in Equation (1) equal to unit length mas of the rail beam, stiffness of the rail beam and Dirac-Delta function respectively. The constant movement velocity of the HST on the rail beam is represented by parameter $v$. The parameter $F$ is the condensed wheelset load. To simplify Equation (1), following parameters are defined by Equation (2)

$$\varepsilon_1^2=\frac{m}{4EI},\qquad \varepsilon_2^4=\frac{k}{4EI} \qquad (2)$$

Equation (1) rearranged as follow using defining given by Equation (2).

$$\frac{\partial^4 w}{\partial x^4}+4\varepsilon_1^2\frac{\partial^2 w}{\partial t^2}+4\varepsilon_2^2 w=-\frac{F}{EI}\delta\left(x-vt\right) \qquad (3)$$

If right side of the equality given by Equation (3) is equal to $F$=0, the homogeneous solution of the Equation (3) is as follows:

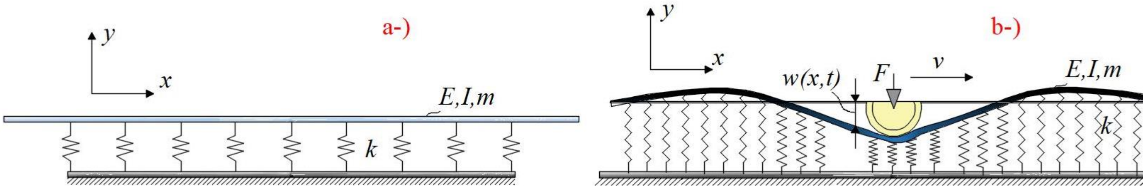

Figure 1 The track beam on elastic foundation, a-) without moving load, b-) with moving load.

$$w\left(x,t\right)=e^{\frac{j2\pi\left(x-ct\right)}{\lambda}} \qquad (4)$$

The parameters $\lambda$ and $c$ in Equation (4) represent vibration wavelength and wave propagation velocity respectively. If Equation (4) rewritten in Equation (3), the following equation is obtained.

$$c=\frac{1}{2\varepsilon_1}\sqrt{\left[\frac{\lambda^2\varepsilon_2^4}{\pi^2}+\left(\frac{2\pi}{\lambda}\right)^2\right]} \qquad (5)$$

When wavelength $\lambda$ is equal to $\frac{\sqrt{2}\pi}{\varepsilon_2}$, the minimum wave propagation velocity is derived by Equation (6).

$$c_{\min}=\sqrt[4]{\frac{4kEI}{m^2}} \qquad (6)$$

The parameter $c_{min}$ in Equation (6) represents minimum wave propagation velocity or critical velocity of bending wave in track structure. The solution of the expression effect of wheelset load given by Equation (3) is $w(x-vt)$. Equation (3) is written as follows by defining $z=x-vt$.

$$\frac{\partial^4 w}{\partial x^4}+4\varepsilon_1^2\frac{\partial^2 w}{\partial t^2}+4\varepsilon_2^2 w=-\frac{F}{EI}\delta\left(z\right) \qquad (7)$$

Characteristic equation for Equation (7) is

$$p^4+4\varepsilon_1^2 v^2 p^2+4\varepsilon_2^4=0 \qquad (8)$$

The solution for the above characteristic equation is related to its coefficient. When $v<\varepsilon_2/\varepsilon_1$ inequality satisfied, the solution of the equation is obtained as follows:

$$p=\pm\alpha\pm j\beta \qquad (9)$$

The expressions in Equation (9) are given by Equation (10).

$$\alpha=\sqrt{\left(\varepsilon_2^2-v^2\varepsilon_1^2\right)},\qquad \beta=\sqrt{\left(\varepsilon_2^2+v^2\varepsilon_1^2\right)} \qquad (10)$$

Then, the solution of the Equation (7) should be obtained by as follows:

$$\begin{aligned}w\left(z\right)=e^{\alpha z}\left(D_1\cos\beta z+D_2\sin\beta z\right)+\\ e^{-\alpha z}\left(D_3\cos\beta z+D_4\sin\beta z\right)+\varphi\left(\left(z\right)\right)\end{aligned} \qquad (11)$$

The parameter $\varphi$(z) in Equation (11) is related to the external moving load $F$ on the rail beam. When $z$ parameter is equal to z=0, in this situation $x$ parameter is to be placed contact point between wheel and rail structure. The four unknown coefficients given by Equation (11) is obtained by following boundary conditions when parameter $z$ is equal to $z$=0.

$$\begin{aligned}&w_1\,|_{z=0}=w_2\,|_{z=0},\quad \frac{\partial w_1}{\partial z}\,|_{z=0}=0,\\ &\frac{\partial w_2}{\partial z}\,|_{z=0}=0,\quad \mathrm{EI}\frac{\partial^3 w_1}{\partial z^3}\,|_{z=0}=\frac{F}{2}\end{aligned} \qquad (12)$$

The solution of the Equation (7) given by following equation.

$$w(z) = -\frac{F}{8EI\alpha\varepsilon_2^2} e^{-\alpha|z|} \left( \cos\beta z + \frac{\alpha}{\beta} \sin\beta\,|z| \right) \qquad (13)$$

## 2.2. Dynamic Analysis of Tack Structure Under Multiple Moving Wheelset

When multiple moving wheelsets is on rail beam, the differential equation given by Equation (3) is derived by following equation.

$$\frac{\partial^4 w}{\partial x^4} + 4\varepsilon_1^2 \frac{\partial^2 w}{\partial t^2} + 4\varepsilon_2^2 w = \ldots$$
$$-\sum_{i=1}^{N} F_i \delta\left(x - a_i - vt\right) \qquad (14)$$

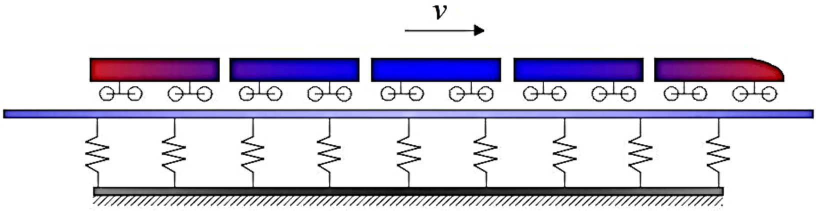

Figure 2 The computational model of a multiple wagon HST.

The parameter $F_i$ in Equation (14) is $i$'th wheelset load, $a_i$ is the distance between first wheelset and $i$'th wheelset, $N$ is total number of wheelsets in HST. The solution of the Equation (14) is obtained by solution of undamped rail beam equation of motion effect of single moving wheelset load given by Equation (13) for each wheelset load separately. Then, these solutions are summed using superposition technique.

## 2.3. Track Equivalent Stiffness and Track Foundation Elasticity Modulus

The track equivalent stiffness parameter $k$ and track foundation elasticity modulus $E_s$ are related to each other. Heelis [16] proposed formulation given by Equation (15) for calculating track equivalent stiffness.

$$k = \frac{0.65E_s}{1 - v_s^2} \sqrt[12]{\frac{E_s B^4}{EI}}, \qquad (15)$$

The parameters $E_s$, $\upsilon_s$, $B$ and $EI$ in Equation (15) are track foundation elasticity modulus, poison ratio, sleeper length and rail flexural modulus respectively.

## 3. NUMERICAL EXAMPLES

In this section, dynamic response of rail beam effect of HST is analysed for different HST, track and foundation parameters. For the track foundation elasticity modulus $E_s$=50 MN/m$^2$ is named as "*Compacted clay*". The track equivalent stiffness for compacted clay is calculated as $k$=56.15 MN/m$^2$ considering poison ratio given in Table 1 $\upsilon_s$=0.35 using Equation (15). The similar values are obtained as $k$=9.82 and 4.63 MN/m$^2$ for loam and soft subgrade respectively considering parameters given in Table 1.

Table 1.
The critical velocity of HST for different track foundation elasticity modulus.

| **Parameters** | **$E_s$ (MN/m$^2$)** | **$k$ (MN/m$^2$)** | **$EI$ (MN/m$^2$)** | **$m$ (kg/m)** | **$B$ (m)** | **$c_{min}$ (m/s)** |
|---|---|---|---|---|---|---|
| **Compacted clay** | 50 | 56.15 | 13.25 | 2735 | 2.5 | 141.24 |
| **Loam** | 10 | 9.82 | 13.25 | 2735 | 2.5 | 91.34 |
| **Soft subgrade** | 5 | 4.63 | 13.25 | 2735 | 2.5 | 75.7 |

As stated in Equation (6) before, the parameter track critical velocity varies depending on track equivalent stiffness, track unit length mass and rail flexural modulus. In this study, sleeper under rail and ballast masses were also taken into account when calculating the track unit length mass. Consequently, the rail unit length mass is calculated as $m$=2735 kg/m as shown in Table 1. Accordingly, the critical velocities are given by Table 1 for three different foundation properties.

### 3.1. Analysis of track vibration under a Moving Wheelset Load

In Figure 3a-c, track displacement caused by a wheelset moving on the track has been plotted for moving wheelset constant velocities $v$=50, 60, 70 m/s considering three different track foundation properties given in Table 1. In these analyses, the moving wheelset velocity is chosen maximum 70 m/s because of the smallest critical velocity value given in Table 1 is smaller than 75.7 m/s which indicated in Table1. Beside this, in these analyses, the constant load value of wheelset moving on the track beam is determined as $F$=170 kN. As shown in Figure 3a-c, maximum rail displacement for certain velocity is obtained in soft subgrade track foundation. Furthermore, the track displacement increases too as the wheelset velocity increases for certain foundation properties

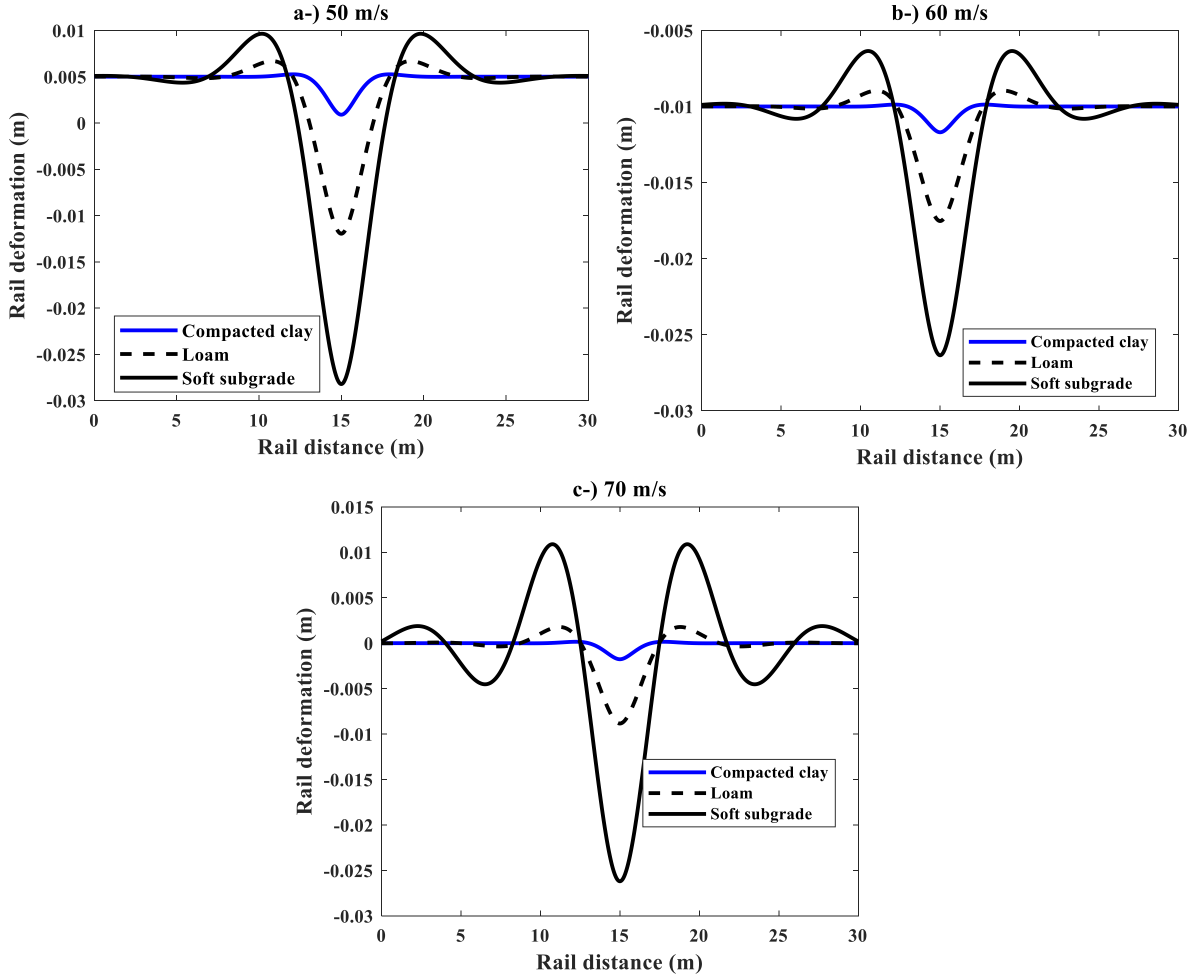

Figure 3 The track displacement for different track foundation and different single train wheelset load.

In Figure 4a-c, the effect of the wheelset velocity upon track displacement is figured out considering three different foundation properties given by Table 1 respectively. As a result of approaching of HST's velocity to rail critical velocity given in Table 1, the rail displacement is excessive increased.

### 3.2. The Track Transverse Vibration Effect of HST

In this section, the track vibration has been analysed under the influence of HST with multiple wheelsets for different train velocity, rail stiffness and track foundation elasticity modulus. Figure 5 shows axle distance for HST and wagon vehicles which used in analysis. In Figure 6a-c, track displacements have been investigated for three different HST's velocities ($v$=50, 60, 70 m/s) moving on the rail considering train wagon number $N$=5 and each wheelset axle load $F$=145 kN. As shown in figure, the maximum rail displacement is determined in soft subgrade track foundation properties given in Table 1 for multiple wheelsets load. Also, it is understood that the most important parameter affecting the rail critical velocity is track foundation elasticity modulus. The rail critical velocity is reduced as shown in Table 1, especially when the track has soft foundation. The critical velocity of soft subgrade foundation given by Table 1 can be easily exceeded by medium-speed train or HST. This case causes excessive vibrations on the rail beam. These vibrations affect riding comfort and riding safety negatively.

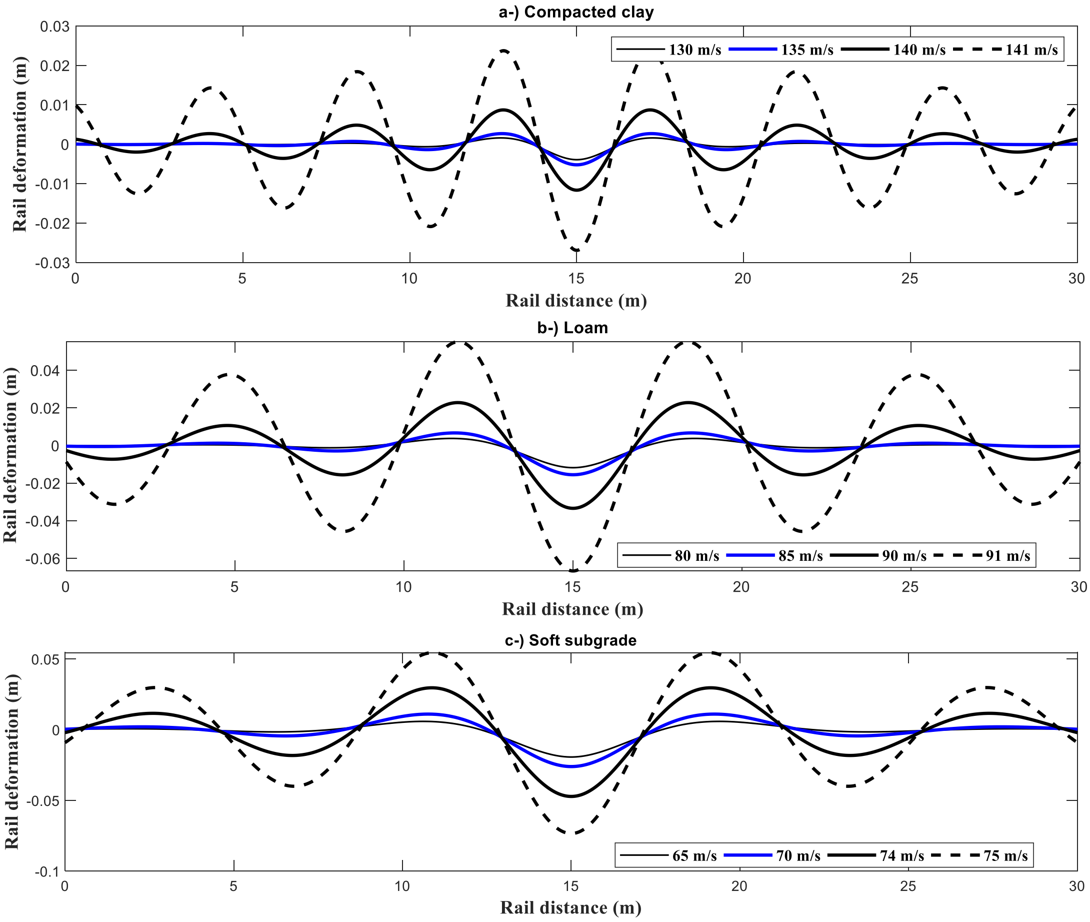

Figure 4 The track displacement for different HST velocity and track foundation.

The rail displacement is indicated for three different wheelset loads ($F$=100, 150, 200 kN) considering compacted clay given by Table 1 as shown in Figures 7a-c. The rail displacement is increased too as shown in figures as the wheelset load is increased. In these analyses, the number of HST's wagon is taken as N=4. In Figures 8a-c, the rail displacement has been analysed for three different rail flexural modulus considering compacted clay track foundation. As shown in figures, the rail displacement has been reduced as the rail flexural modulus is increased.

Moreover, it is understood that rail flexural modulus doesn't affect rail displacement as shown in Figures 8a-c when moving HST's velocity is smaller than rail critical velocity. But the rail displacement is significantly reduced as rail flexural modulus is increased when HST's velocity approaches the rail critical velocity.

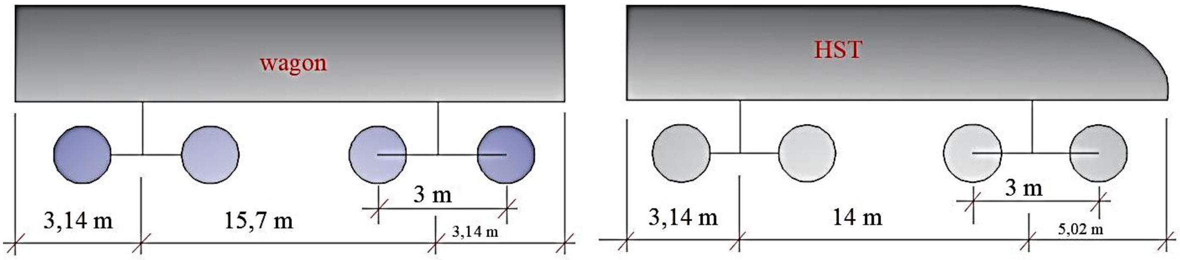

Figure 5. The computational model of a multiple wagon HST.

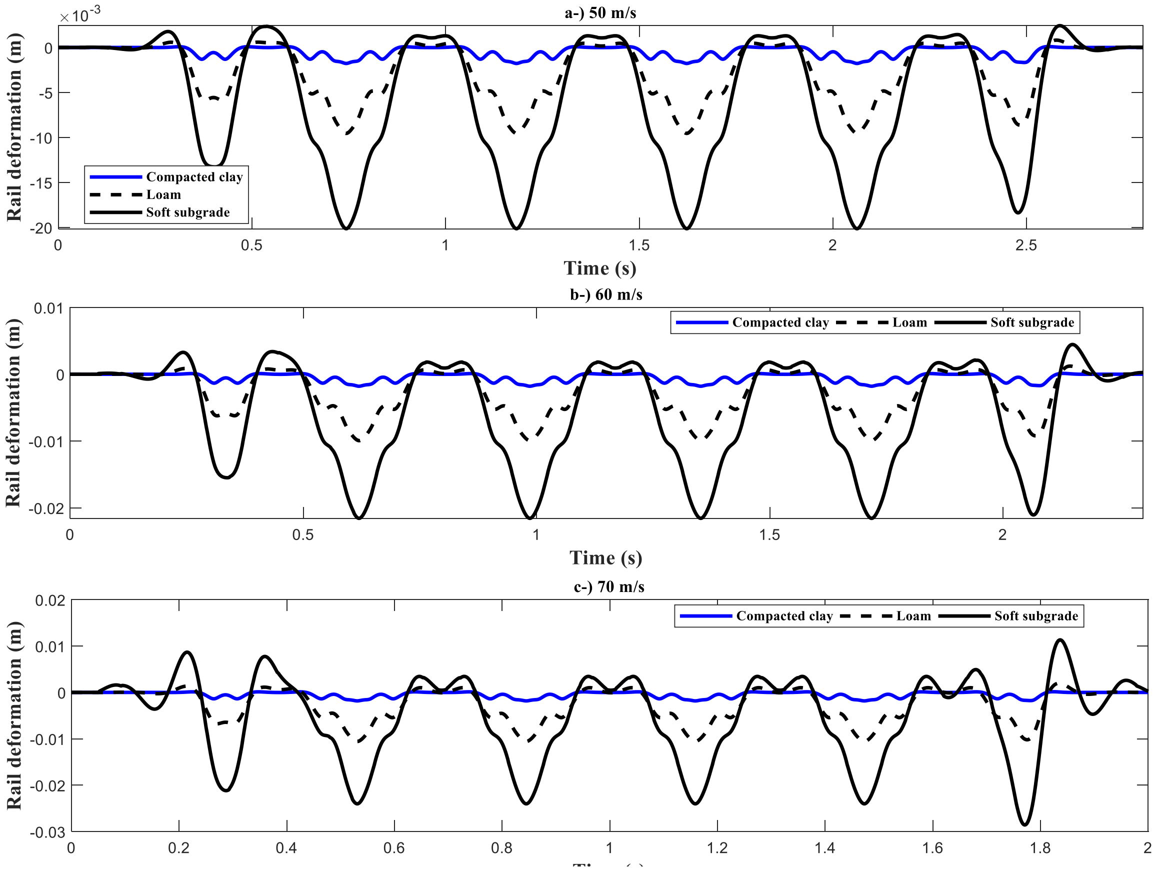

Figure 6 The track displacements considering various track foundation for the three different train velocities.

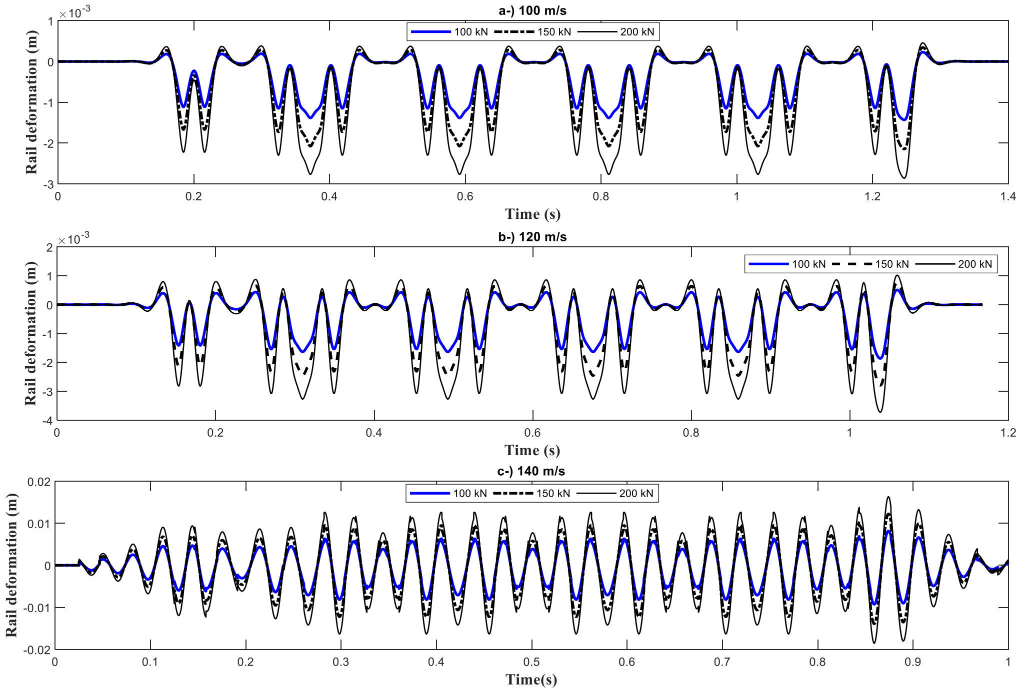

Figure 7 The track displacements considering various wheel load foundation for the three different train velocities.

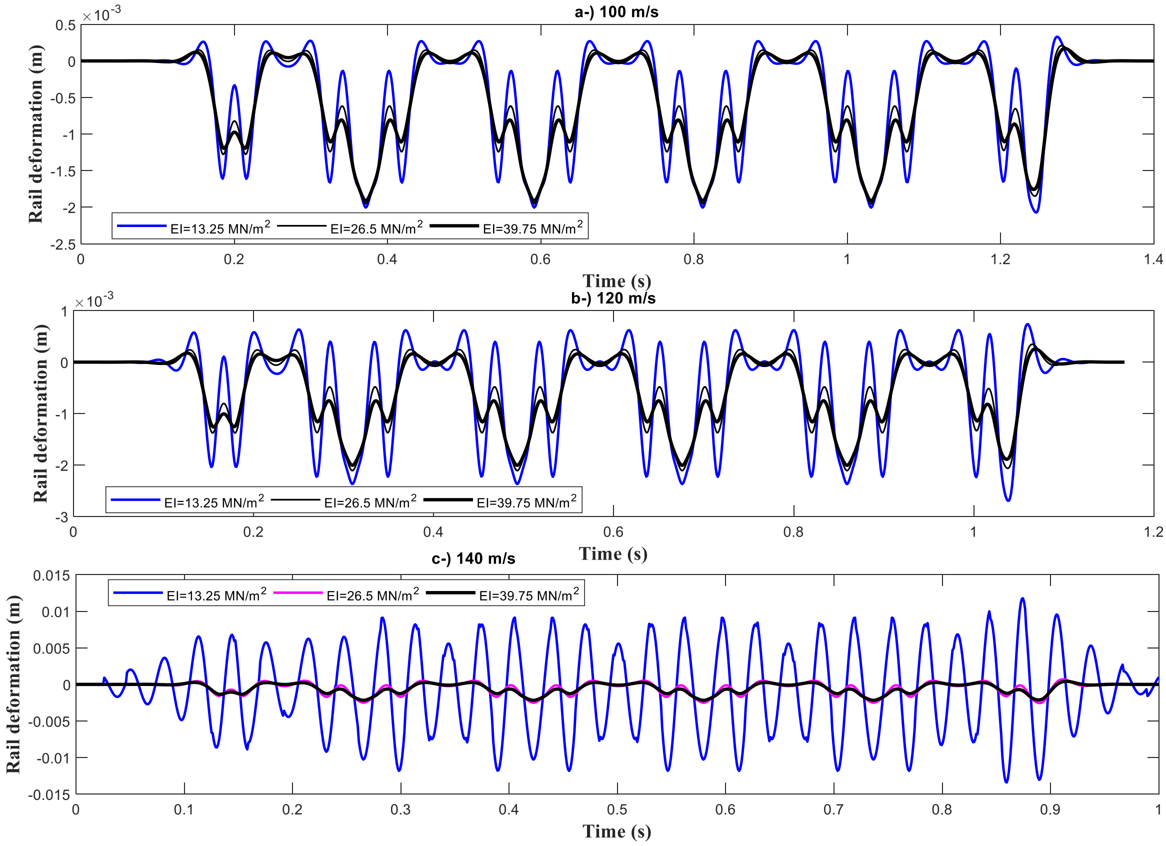

Figure 8 The track displacements considering various track foundation rigidity for the three different train velocities.

Figure 8 shows that rail stiffness has a great importance on the rail vibration waveform and its amplitude. When the rail stiffness is reduced, the waveform of vibration is multiple and vibration amplitude is large vice versa. Likewise, the rail stiffness is increased, the vibration wave number is reduced and vibration amplitude is decreased. This phenomenon is not clear when HST's velocity is very smaller than rail critical velocity as shown in figures. However, when the HST's velocity approaches the rail critical velocity, it is seen very clearly in Figure 8c. On the other hand, the vibration amplitude is increased too as the vibration wave frequency is raised. Consequently, this case causes liquefaction of ballast material under rail which would reduce the stability of the track foundation.

## 4. CONCLUSIONS

In this study, the rail which most important structure of railway transportation is modelled analytically according to elastic beam theory considering track foundation. Then, the effects of parameters such as ballast elasticity modulus, rail stiffness and rail critical velocity have been investigated in detail under influence of moving HST. Based on these computational results and considering three different track foundations, some conclusions are summarized as follows:

-The amplitude of the displacement on the rail beam is increased as the elasticity modulus of the ballast material is decreased.

-The rail displacement is gradually increased as the velocity of the HST moving on the rail beam approaches the rail critical velocity.

-As the wheelset axle load moving on the rail is increased, the rail displacement of the rail beam is increased too.

-The rail displacement is significantly decreased because of the rail flexural modulus is raised. This situation has been seen clearly especially HST's velocity approaches to rail critical velocity. The vibration wave amplitude increases as the train speed approaches the rail infrastructure critical speed.

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## REFERENCES

[1] S.L. Knothe, K.L; Grasssie, Modelling of railway track and vehicle/track interaction at high frequencies, Veh. Syst. Dyn. 22 (1993) 209–262.

[2] Z.P. Zeng, Z.W. Yu, Y.G. Zhao, W.T. Xu, L.K. Chen, P. Lou, Numerical simulation of vertical random vibration of train-slab track-bridge interaction system by PEM, Shock Vib. 2014 (2014).

doi:10.1155/2014/304219.

[3] Y.H. Chen, Y.H. Huang, C.T. Shih, Response of an infinite tomoshenko beam on a viscoelastic foundation to a harmonic moving load, J. Sound Vib. 241 (2001) 809–824. doi:10.1006/jsvi.2000.3333.

[4] B. Biondi, G. Muscolino, A. Sofi, A substructure approach for the dynamic analysis of train-track-bridge system, Comput. Struct. 83 (2005) 2271–2281. doi:10.1016/j.compstruc.2005.03.036.

[5] E. Celebi, G. Schmid, Investigation of ground vibrations induced by moving loads, Eng. Struct. 27 (2005) 1981–1998. doi:10.1016/j.engstruct.2005.05.011.

[6] M. Hetenyi, Beams on elastic foundation: Theory with applications in the fields of civil and mechanical engineering, University of Michigan Press, 1946.

[7] D.G. Duffy, The response of an infinite railroad track to a moving, vibrating mass, J. Appl. Mech. 57 (1990) 66–73.

[8] S.L. Grassie, S.J. Cox, The Dynamic Response of Railway Track With Flexible Sleepers to High Frequency Vertical Excitation, Proc. Inst. Mech. Eng. , Part I J. Syst. Control Eng. 198 (1984) 117–124. doi:10.1243/PIME.

[9] S. Iwnicki, Handbook of railway vehicle dynamics, 2006. doi:10.1201/9781420004892.

[10] A.P. De Man, Pin-pin resonance as a reference in determining ballasted railway track vibration behaviour, Heron. 45 (2000) 35–52.

[11] D.J. Thompson, Railway noise and vibration: mechanisms, modelling and means of control., Elsevier Science, Oxford, 2009.

[12] S.L. Grassie;, R.W. Gregory;, K.L. Johnson;, The dynamic response of railway track to high frequency vertical excitation., J. Mech. Eng. Sci. 24 (1982) 77–90.

[13] G.A. Hunt, Dynamic analysis of railway vehicle/track interaction forces, Loughborough University of Technology, 1986.

[14] W. Zhai, Vehicle – Track Coupled Dynamics, n.d.

[15] L. Fryba, Vibration solids and structures under moving loads, Thomas Telford House, 1999.

[16] M.E. Heelis, A.C. Collop, A.R. Dawson, D.N. Chapman, V.V. Krylov, Transient effects of high speed trains crossing soft soil, in: 12th Eur. Conf. Soil Mech. Geotechincal Eng., Amsterdam, 1999: pp. 1809–1814.