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Multistability and Coexisting Attractors in a New Circulant Chaotic System

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In this paper, a new four-dimensional chaotic flow is proposed. The system has a cyclic symmetry in its structure and shows a complicated, chaotic attractor. The dynamical properties of the system are investigated. The system shows multistability in an interval of its parameter. Fractional

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order model of the proposed system is discussed in various fractional orders. Bifurcation analysis of the fractional order system shows that it has a kind of multistability like the integer order system, which is a very rare phenomenon. Circuit realization of the proposed system is also carried out to show that it is usable for engineering applications.

Keywords: Chaotic flow; bifurcation; multistability; fractional order; circuit realization.

1. Introduction

Chaos is an important phenomenon in nonlinear dynamical systems. Many systems in different fields such as biology and economics exhibit chaos, and the study of such systems has progressed in recent decades. Proposing new chaotic systems with desired properties has been a hot topic in recent years. Many of those systems are with special structural features. For example, we can point out chaotic systems with only stable equilibria [Molaie *et al.*, 2013], with curves of equilibria [Barati *et al.*, 2016], with planes or surfaces of equilibria [Jafari *et al.*, 2016b; Jafari *et al.*, 2016a], with no equilibrium points [Jafari *et al.*, 2013; Rajagopal *et al.*, 2017b; Wei, 2011], and with nonhyperbolic equilibria [Wei *et al.*, 2015a; Wei *et al.*, 2015b]. From other points of view, chaotic systems with multi-scroll attractors [Munoz-Pacheco *et al.*, 2014; Tlelo-Cuautle *et al.*, 2015], multistability [Lai & Chen, 2016a; Kengne *et al.*, 2017; Sharma *et al.*, 2015; Kapitaniak & Leonov, 2015], extreme multistability [Bao *et al.*, 2016a; Bao *et al.*, 2016b; Bao *et al.*, 2017b; Bao *et al.*, 2017a], megastability [He *et al.*, 2018; Khalaf *et al.*, 2018; Wei *et al.*, 2018; Sprott *et al.*, 2017], hidden attractors [Dudkowski *et al.*, 2016; Leonov *et al.*, 2011, 2012; Leonov & Kuznetsov, 2013; Brezetskyi *et al.*, 2015], and with both self-excited and hidden chaotic attractors [Jafari *et al.*, 2017; Rajagopal *et al.*, 2017d; Rajagopal *et al.*, 2017a] have been studied.

Multistability is an important topic in nonlinear dynamics [Jaros *et al.*, 2016; Lai & Chen, 2016b; Lai *et al.*, 2017; Pan & Li, 2017]. In some occasions multistability is unwanted, while in some other cases it is desired. Sometimes it is essential for a dynamical system to work in a specific attractor and not go out from it. In such a situation, multistability is a potential danger (because due to any disturbance the system can go to a new unwanted attractor). On the other hand, multistability makes systems flexible without tuning parameters [Pisarchik & Feudel, 2014]. In any case, a better understanding of such systems is interesting. The main feature of multistable systems is having coexisting attractors [Pham *et al.*, 2018, 2017]. Some cases of multistable chaotic systems such as systems with mega-stability have been presented in recent years [Sprott *et al.*, 2017]. Fractional order systems are more complex than integer orders. So it is important to investigate multistabilities in fractional order systems that may cause a potential danger in the function of systems.

In this paper, we propose a new chaotic system with cyclic symmetry. Such systems are called circulant [Sprott, 2010]. The proposed system shows multistability and coexisting attractors which are investigated in detail. Also, fractional order model of the proposed system is derived and studied. Interestingly, the system shows multistability with variations in fractional order which is very rare in the literature. Also the system is investigated through a circuit design procedure.

The novelty of the proposed system is the nature of the system's equations which shows cyclic symmetry. Such a system is rare in literature apart from the well-known Thomas cyclic symmetry system [Thomas, 1999] and some of our earlier works [Gugapriya *et al.*, 2019; Rajagopal *et al.*, 2018]. We quote the exact words from the article [Sadooghi, 2018] which says "Structural systems with cyclic symmetry are an important class of engineering applications, which are composed of some identical substructures. They commonly use in large circular space antennas, bladed disk assemblies, and magnetic storage devices". In [Sarrouy *et al.*, 2011] the authors have discussed the free and forced vibrations of a nonlinear cyclic symmetry structure. In [Grolet & Thouverez, 2012] a similar discussion was made on cyclic symmetry systems with geometric nonlinearities. In the comparison of the proposed system with those already in the literature, the proposed system shows multistability and coexistence of attractors.

2. Cyclic Symmetry Chaotic System (CSC)

Chaotic systems with symmetry in their equations are interesting. Cyclic or ring symmetry is an example of symmetry in which the state variables are arranged in a ring of elements each connected to its nearest state variable in an identical pattern. In this paper, we propose a novel 4D system with cyclic symmetry which is derived from [Thomas, 1999],

$$\begin{aligned}\dot{x} &= f(x, y, z, w),\\ \dot{y} &= f(y, z, w, x),\\ \dot{z} &= f(z, w, x, y),\\ \dot{w} &= f(w, x, y, z).\end{aligned} \tag{1}$$

Choosing $f(x, y, z, w) = ax + by - y^3$, Eq. (1) is modified to a new 4D system defined as,

$$\begin{aligned}\dot{x} &= ax + by - y^3,\\ \dot{y} &= ay + bz - z^3,\\ \dot{z} &= az + bw - w^3,\\ \dot{w} &= aw + bx - x^3,\end{aligned} \tag{2}$$

where $a = -3$, $b = 9$. These parameters are obtained by searching for chaotic attractors in System (2).

The proposed CSC system is dissipative. To prove this, let us define Ω to be any region in R^4 with a smooth boundary and also, $\Phi(t) = \Phi_t(\Omega)$, where Φ_t is the flow of the vector field f. Furthermore, let $V(t)$ denote the hypervolume of $\Phi(t)$. Based on Liouville's theorem [Gibbs, 1884], we have

$$\dot{V} = \int_{\Phi(t)} (\nabla \cdot f) dxdydzdw. \tag{3}$$

The divergence of the vector field f is easily calculated as

$$\begin{aligned}\nabla \cdot f &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w}\\ &= a + a + a + a = -12.\end{aligned} \tag{4}$$

Substituting (4) into (3), we obtain the first order differential equation

$$\dot{V}(t) = -12V(t). \tag{5}$$

Integrating (5), we obtain the unique solution as

$$V(t) = \exp(-12t)V(0) \tag{6}$$

for all $t \geq 0$ which confirms that $V(t) \to 0$ exponentially when $t \to \infty$. So the CSC system is dissipative. Figure 1 shows the phase portraits of the CSC system for two symmetric initial conditions.

System (2) has 81 equilibrium points and origin "O" is one of them. Eigenvalues of system (2) at "O" are given by

$$\lambda_{1,2} = \pm\sqrt{(a^2 + b^2)},$$

$$\lambda_{3,4} = \sqrt{\pm((a + b)(a - b))}$$

and its characteristic polynomial is $\lambda^4 - 2a^2\lambda^2 + a^4 - b^4$. Eigenvalues for $a = -3$, $b = 9$ are ± 9.4868, $\pm 8.4853i$. So this equilibrium is not stable. Lyapunov Exponents (LEs) of the system which are calculated using Wolf's algorithm [Wolf *et al.*, 1985] are $L_1 = 0.8529$, $L_2 = 0$, $L_3 = -4.261$, $L_2 = -8.592$.

To investigate the impact of the parameters on the proposed CSC system, bifurcation diagram is investigated. Parameter b is considered as the bifurcation parameter which shows different dynamics in the studied interval. Figure 2 shows the bifurcation diagram of the CSC system and its corresponding maximum Lyapunov exponent (MLE). The CSC system shows multiple chaotic regions from $b = 8.11$ to $b = 9.844$. The first chaotic region is seen in $8.11 \leq b \leq 8.168$ and it follows by multiple chaotic regions.

The proposed CSC system shows possibilities of hysteresis and the existence of bistability as can be seen in Fig. 3. The bifurcation diagram is obtained by plotting local maxima of the coordinate x in terms of increasing (or decreasing) parameter in tiny steps in the interval $8.5 \leq b \leq 9.5$ as shown in Fig. 3(a). The final state at each parameter serves as the initial state for the next parameter. This strategy, known as forward and backward continuation, represents a simple way to localize the parameter interval in which the system develops multistability. The existence of multistability can be confirmed by comparing the forward [Fig. 3(a), red] and backward [Fig. 3(a), black] bifurcation diagrams. To see the multistable plots clearly, we derived the bifurcation plots for a small window of $8.9 \leq b \leq 9.1$ which is shown in Fig. 3(b). Figure 4 shows the maximum Lyapunov exponent (MLE) of the CSC system with respect to changing parameter b. The red plot in Fig. 4 shows the MLE when the parameter is increased from min to max and black plot shows when the parameter is decreased. Figure 5 shows

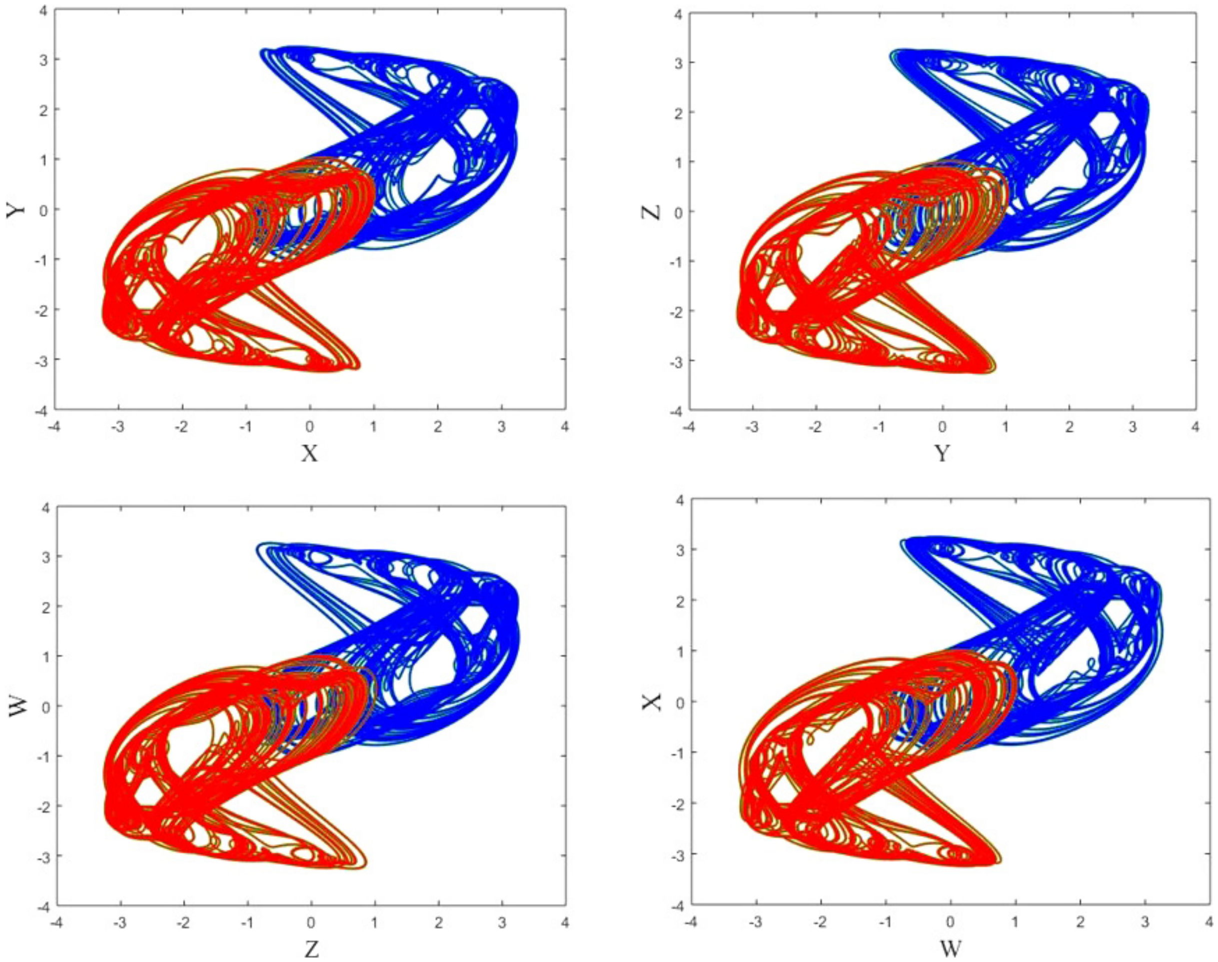


Fig. 1. 2D phase portraits of the CSC system. The blue plot shows the phase portrait with initial conditions $[0.4, 0.1, 0.1, 0]$ and the red plot shows the phase portrait with initial conditions $[-0.4, -0.1, -0.1, 0]$.

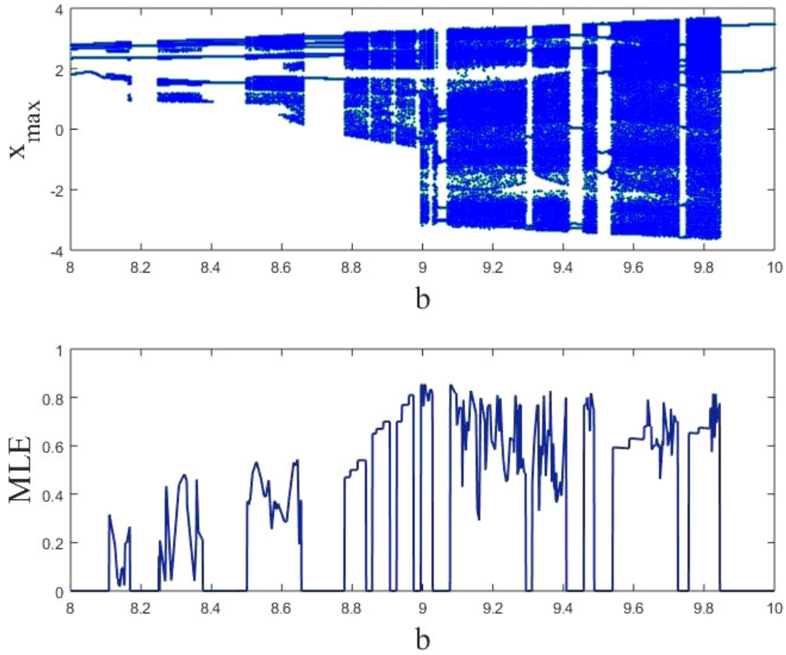


Fig. 2. Bifurcation diagram and maximum Lyapunov exponent of the CSC system with respect to changing parameter b.

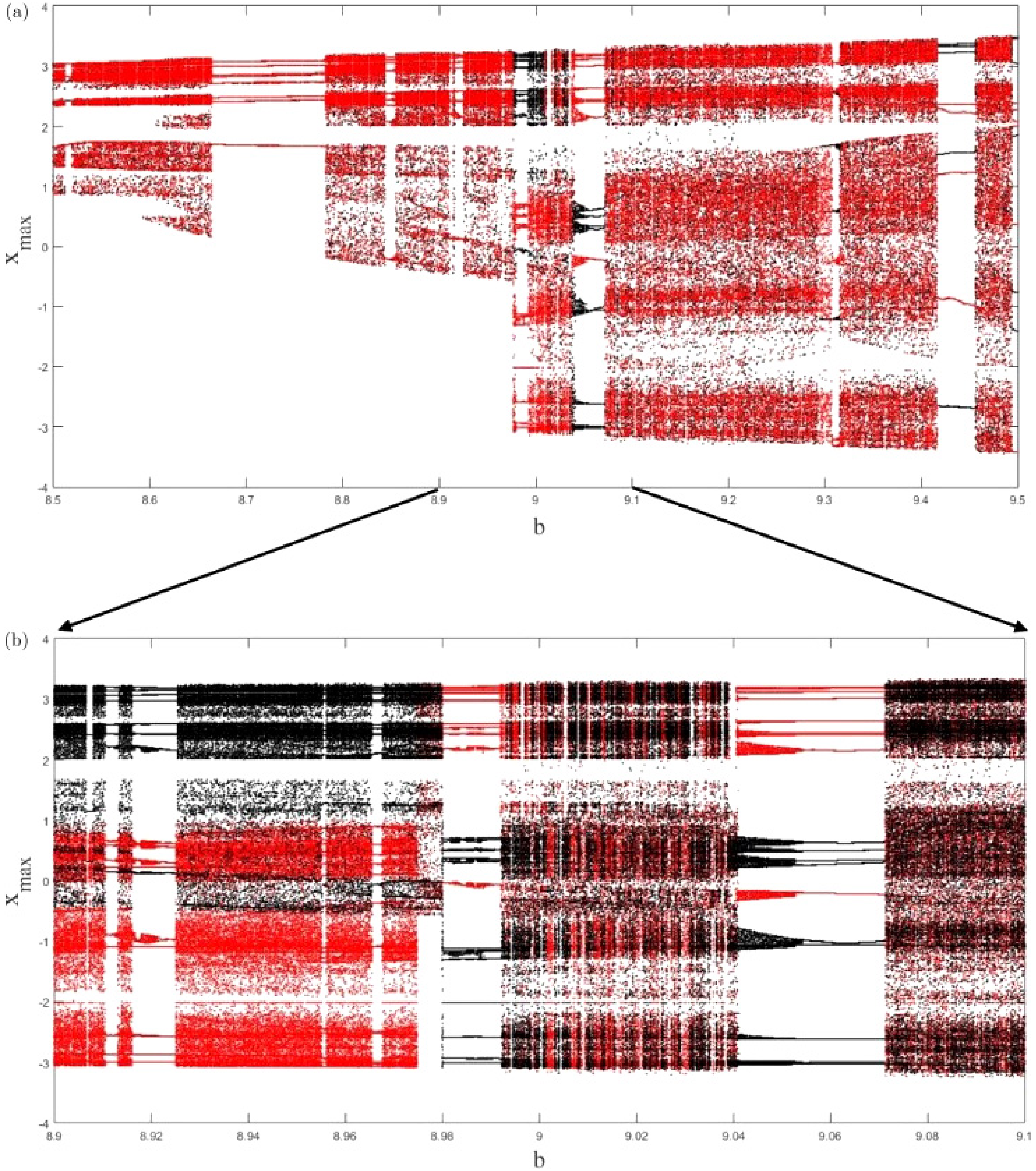


Fig. 3. Bifurcation diagram of CSC system with forward continuation in red color and backward continuation in black color. (a) Multistability of the CSC system with respect to changing parameter b and (b) zoomed view of the multistable region.

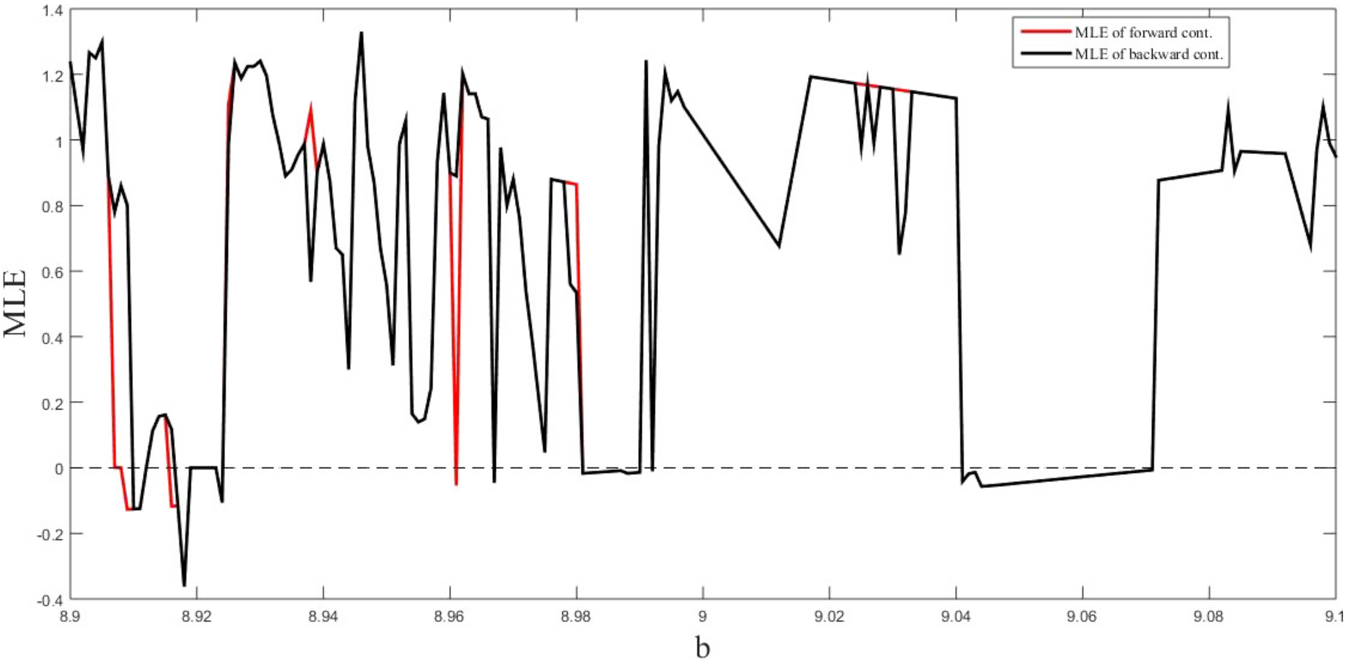
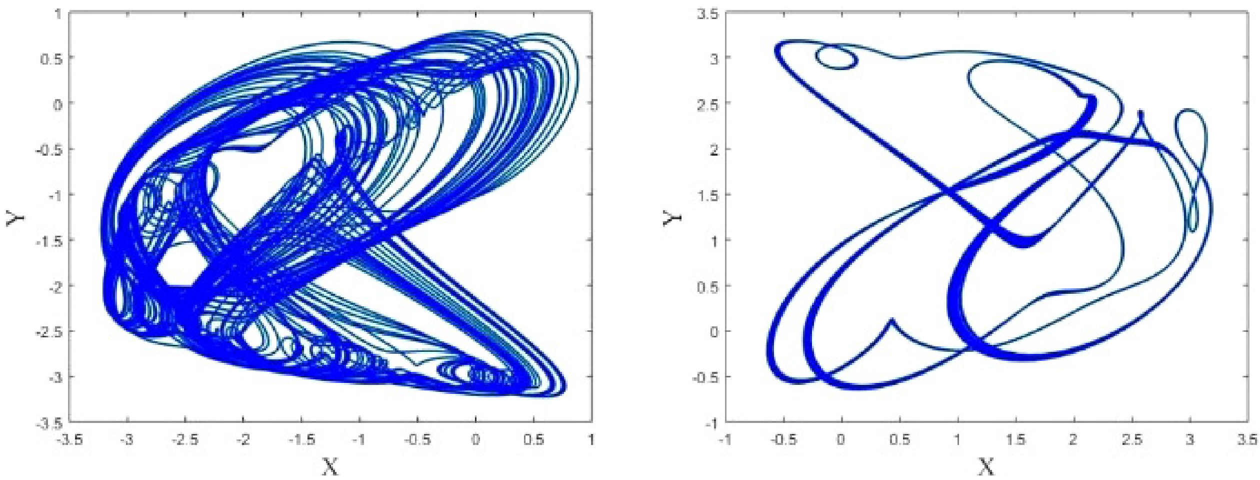


Fig. 4. Maximum Lyapunov exponent (MLE) of the CSC system with respect to changing parameter b, forward continuation in red color and backward continuation in black color.



(a) (b)

Fig. 5. 2D phase portraits of the CSC system for (a) $b = 8.9$, (b) $b = 8.92$, (c) $b = 8.989$, (d) $b = 9$, (e) $b = 9.042$, (f) $b = 9.06$, (g) $b = 9.08$ and (h) 9.5.

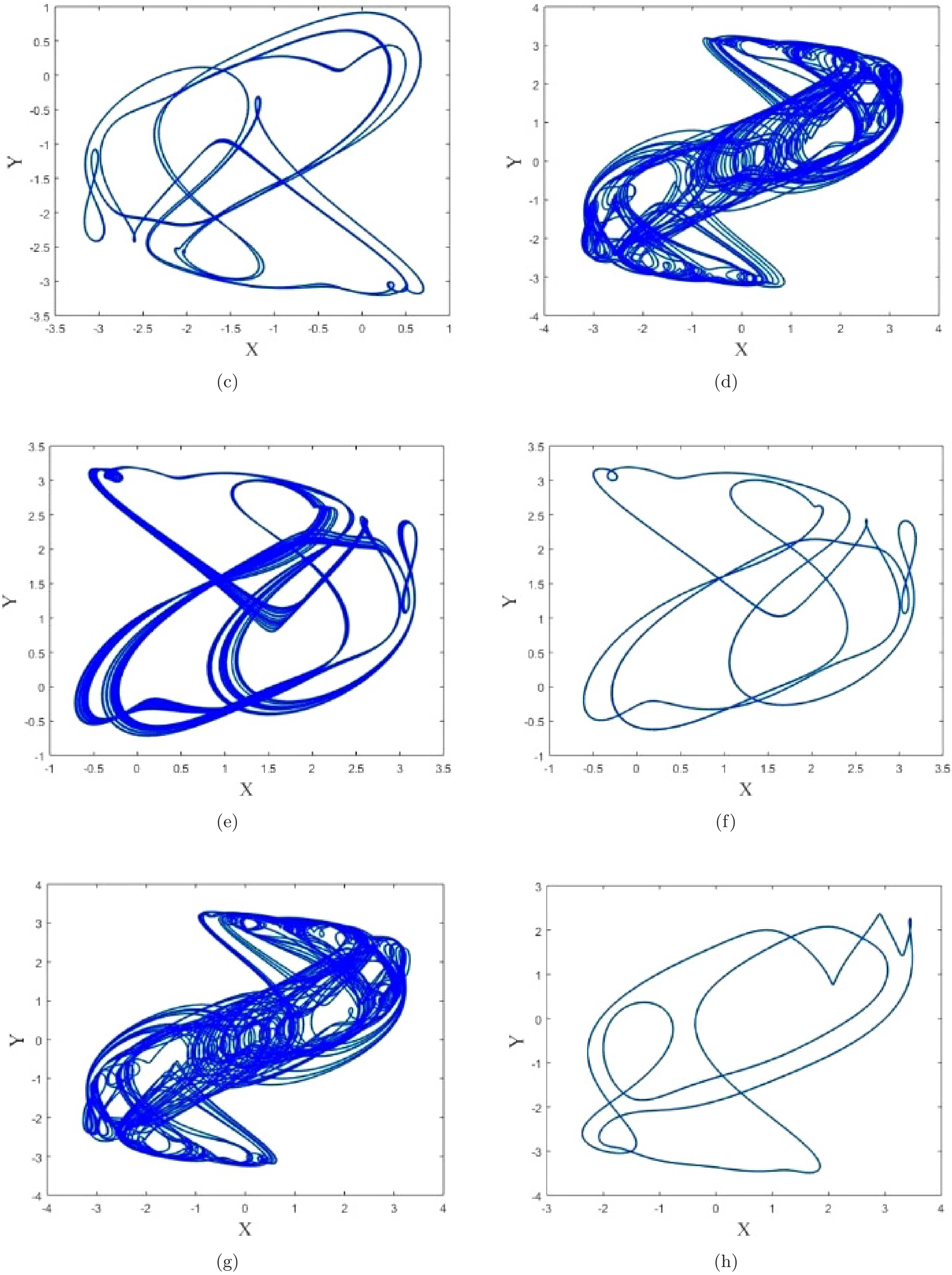


Fig. 5. (*Continued*)

2D projection of attractors of the CSC system for changing parameter b.

3. Fractional Order Cyclic Symmetry Chaotic System (FOCSC)

Fractional calculus has a history dating back to the 17th century, but it found its applications in science and engineering only in recent years. Many physical systems such as dielectric polarization, electromagnetic waves, and quantum evolution of complex systems exhibit fractional order dynamics. Thus fractional order control algorithms are attracting the attention of researchers [Hilfer, 2000; Rajagopal *et al.*, 2017e; Rajagopal *et al.*, 2017c]. Recently, many fractional order chaotic and hyperchaotic systems have been discussed in the literature [Rajagopal *et al.*, 2017b; Rajagopal *et al.*, 2017d; Rajagopal *et al.*, 2017a; Lakshmikantham & Vatsala, 2008].

In this section, we derive the fractional order cyclic symmetry system (FOCSC) using the Caputo fractional derivative [Diethelm, 1997] defined by,

$$D^q x = J^{m-q} x^m, \tag{7}$$

where m is the integer closest to q with $m > q$ and J^α is the αth order Riemann–Liouville integral operator given by

$$J^\alpha y = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \tag{8}$$

where Γ is the gamma operator.

Using the qth order Caputo fractional derivative [(7) and (8)], the FOCSC system is derived as,

$$\begin{aligned} D^{q_x} x &= ax + by - y^3, \\ D^{q_y} y &= ay + bz - z^3, \\ D^{q_z} z &= az + bw - w^3, \\ D^{q_w} w &= aw + bx - x^3, \end{aligned} \tag{9}$$

where $a = -3$, $b = 9$ are the parameters of the system.

For the numerical analysis of the FOCSC system, there are three main approaches, namely frequency-domain method [Charef *et al.*, 1992], Adomian Decomposition Method (ADM) [Adomian, 1990] and Adams–Bashforth–Moulton (ABM) algorithm [Diethelm, 1997, 2010]. The frequency-domain method is not always reliable in detecting chaotic dynamics of nonlinear systems [Tavazoei & Haeri, 2007]. In this section we use the predictor–corrector method of Adams–Bashforth–Moulton (ABM) studied in [Diethelm, 2010] whose convergence and accuracy are high as discussed in [Diethelm & Freed, 1998]. The predictor–corrector approach for the FOCSC system can be derived as discussed in [Diethelm, 2010]. The entire numerical methods to solve the FOCSC system is detailed in the Appendix. The 2D phase portraits of the FOCSC system solved with ABM are shown in Fig. 6.

Dynamical properties of the proposed FOCSC such as the equilibrium points, eigenvalues, characteristic equation, Lyapunov exponents, bifurcation plots and multistability feature are discussed in this section.

3.1. *Stability of equilibrium*

As mentioned in Sec. 2, the CSC system (2) has 81 equilibrium points and origin "O" is one of them. Similarly, the FOCSC system has 81 equilibrium points including the one at origin. We focus our discussion on the stability of the equilibrium at "O". The discussion on the stability of the equilibrium point can be with reference to the fractional order q of FOCSC. We have used the methods discussed in [Rajagopal *et al.*, 2017b; Tavazoei & Haeri, 2009] to derive the stability conditions for the FOCSC system.

Commensurate Order: The condition in which FOCSC is asymptotically stable is $|\arg(\lambda)| > \frac{q\pi}{2}$ for all eigenvalues λ. In critical eigenvalues, the system is stable if $|\arg(\lambda)| \geq \frac{q\pi}{2}$ where the critical eigenvalue of $|\arg(\lambda)| = \frac{q\pi}{2}$ has geometric multiplicity of one. In other words, the condition in which FOCSC system's eigenvalues have positive real parts is $q > \frac{2}{\pi}\tan^{-1}(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)})$. The characteristic equation of the system is calculated by $\det(\mathrm{diag}[\lambda^{Mq_x}, \lambda^{Mq_y}, \lambda^{Mq_z}, \lambda^{Mq_w}] - J_E) = 0$. The characteristic equation in $q = 0.99$ is $\lambda^{396} + 4\lambda^{298} + 6\lambda^{200} - 2a^2\lambda^{198} + 4\lambda^{102} - 4a^2\lambda^{100} + \lambda^4 - 2a^2\lambda^2 - (b^4 - a^4) = 0$. In $a = -3$, $b = 9$ the equation is $\lambda^{396} + 4\lambda^{298} + 6\lambda^{200} - 18\lambda^{198} + 4\lambda^{102} - 36\lambda^{100} + \lambda^4 - 18\lambda^2 - 6480 = 0$. Using the Matlab "roots" function, we can verify that not all the eigenvalues of the equation satisfy the condition $|\arg(\lambda)| > \frac{q\pi}{2}$

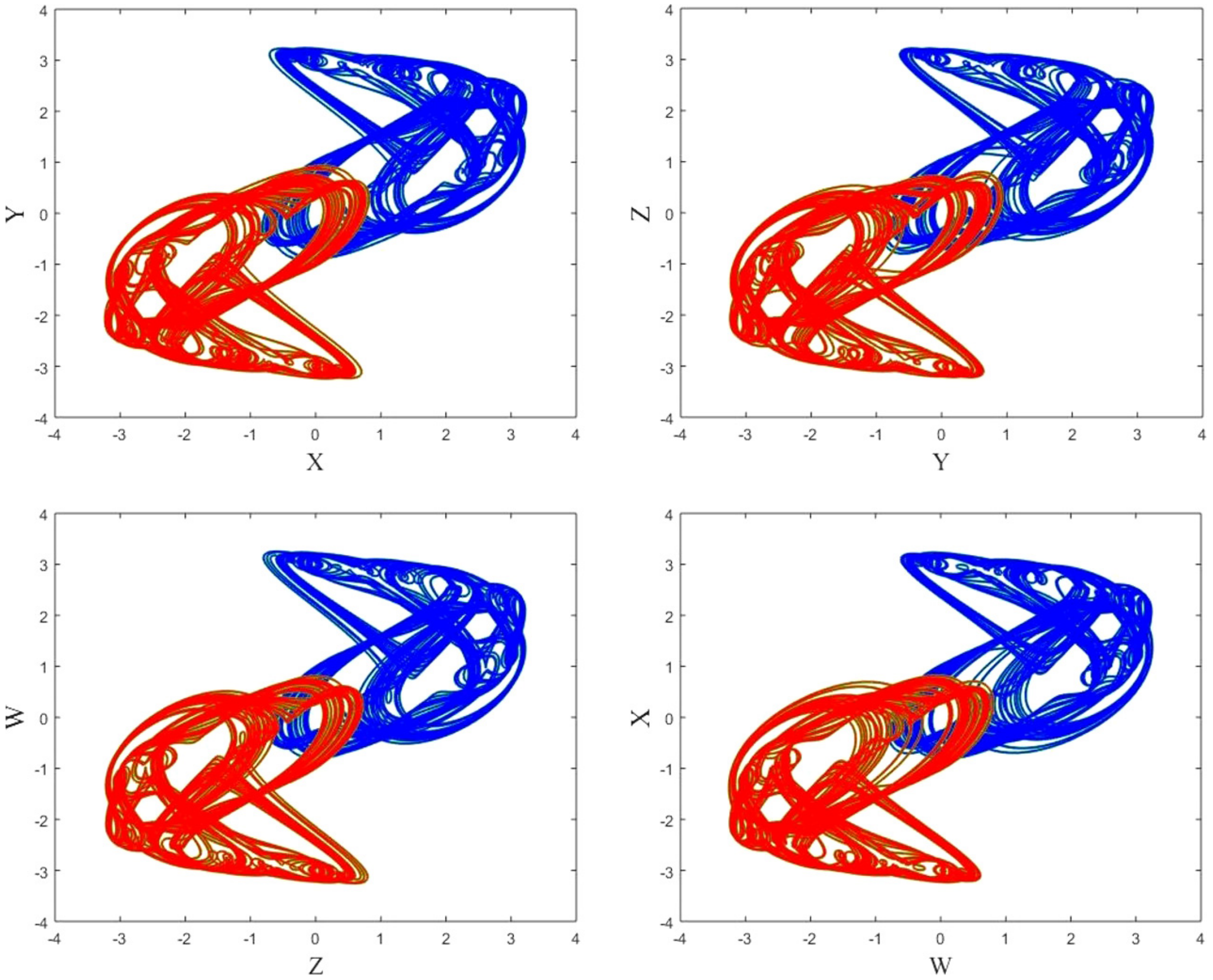


Fig. 6. 2D phase portraits of the FOCSC system with fractional order $q = 0.99$. The blue plot shows the phase portrait with initial conditions $[0.4, 0.1, 0.1, 0]$ and the red plot shows the phase portrait with initial conditions $[-0.4, -0.1, -0.1, 0]$.

for $q \geq 0.89$. Thus they make the equilibrium unstable [Rajagopal *et al.*, 2017b].

Incommensurate Order: Considering that the fractional order of the FOCSC is incommensurate $q_i = [q_x, q_y, q_z, q_w]$, $q_i = \frac{num(i)}{den(i)}$ and $gdc(num(i), den(i)) = 1$ for $i = [x, y, z, w]$. Where "M" is $LCM(den(i))$, then the system is globally asymptotically stable if all the eigenvalues of the system obey $\frac{q\pi}{2M} - \min_i\{\arg(\lambda_i)\} \geq 0$. In other words, $q > \frac{2M}{\pi}\min_i\{\arg(\lambda_i)\}$ which is not satisfied for all values of $q > 0.79$. Hence as in [Rajagopal *et al.*, 2017b; Petras, 2011; Rivero *et al.*, 2013], we could confirm that the equilibrium "O" is unstable.

3.2. *Lyapunov exponents and dissipativity*

The Lyapunov exponents (LEs) of the FOCSC system are calculated using the algorithm in [Danca & Kuznetsov, 2018]. LEs are $L_1 = 0.8825$, $L_2 = 0$, $L_3 = -4.487$, $L_4 = -8.613$ in $q = 0.99$. The fde12 (predictor–corrector method) along with the Wolf algorithm as used in [Danca & Kuznetsov, 2018] are applied to calculate the LEs of FOCSC system. The fractional order systems sometimes give the advantage of more complex dynamics (higher MLE) in comparison with their integer order counterpart. The MLE of FOCSC is higher than CSC system. It could be easily noted that the sum of the LEs is negative, confirming that the FOCSC is dissipative.

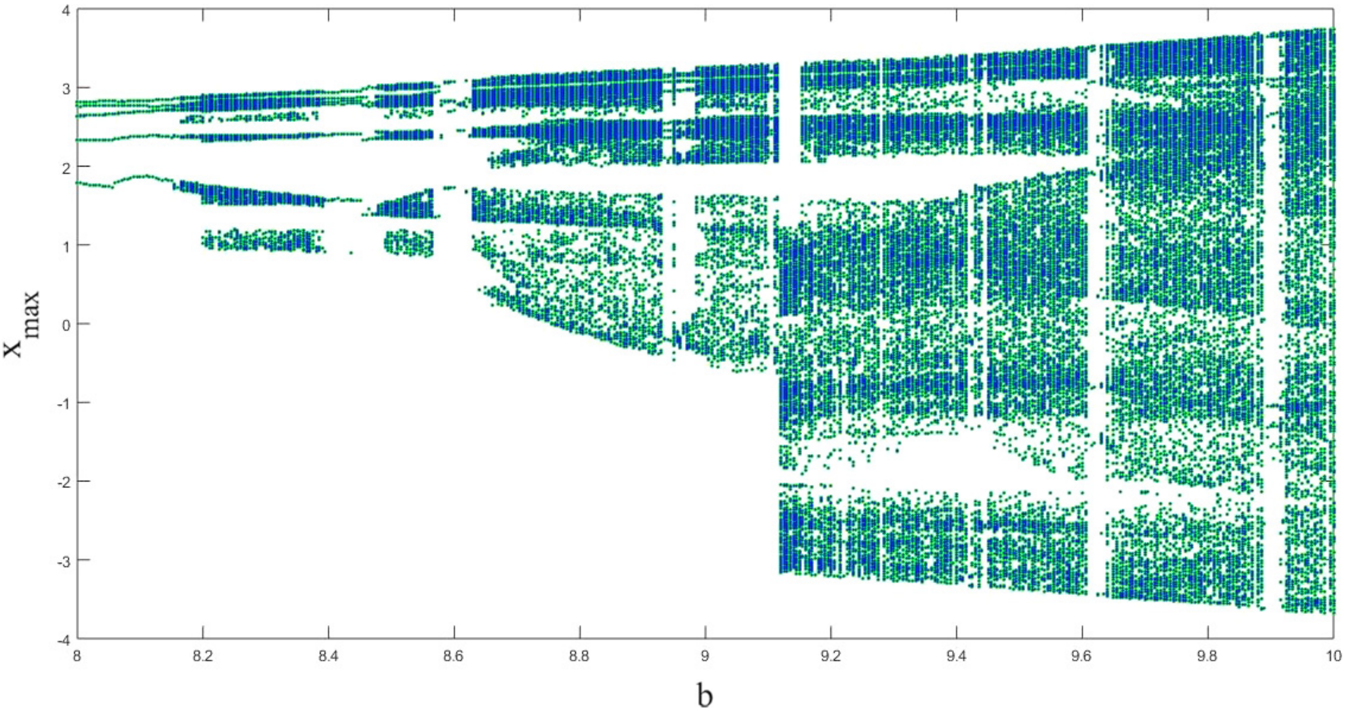


Fig. 7. Bifurcation diagram of the FOCSC system with respect to changing parameter b for commensurate fractional order $q = 0.99$ and initial conditions $[0.4, 0.1, 0.1, 0]$.

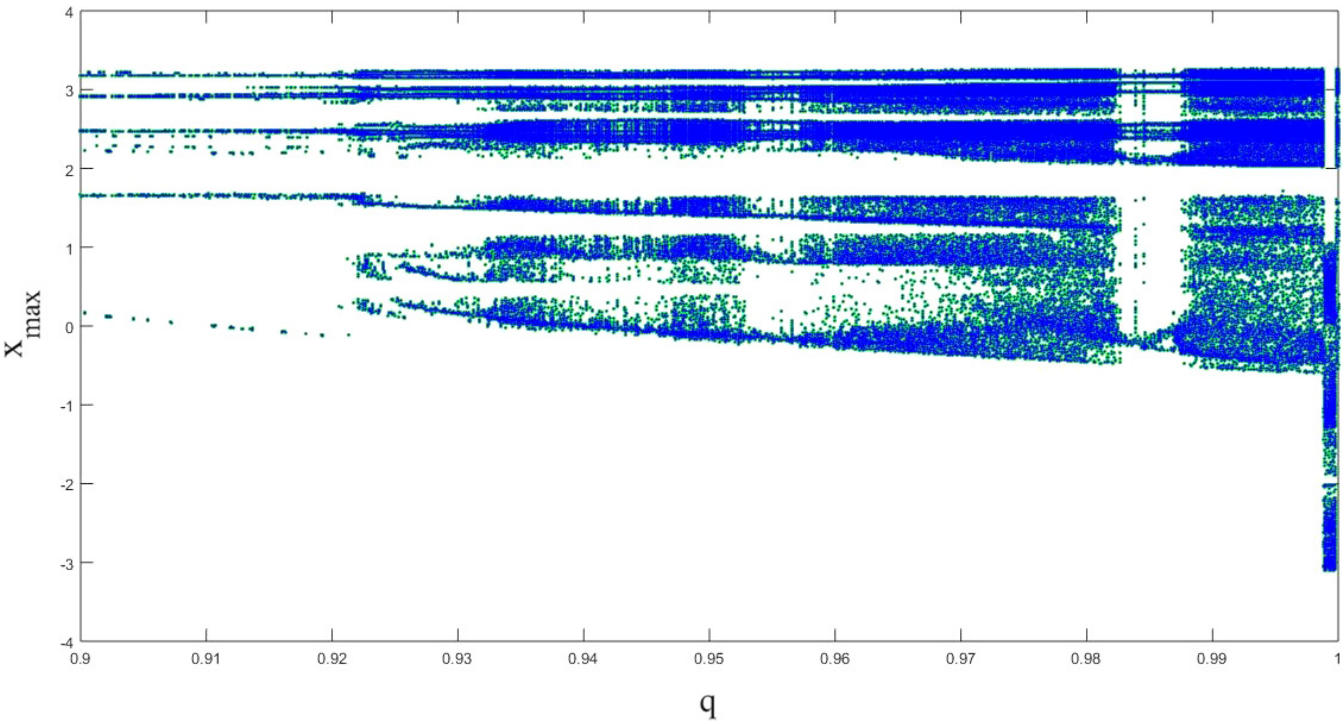


Fig. 8. Bifurcation diagram of FOCSC system with respect to changing commensurate fractional order q, constant parameter $b = 9$ and initial conditions $[0.4, 0.1, 0.1, 0]$.

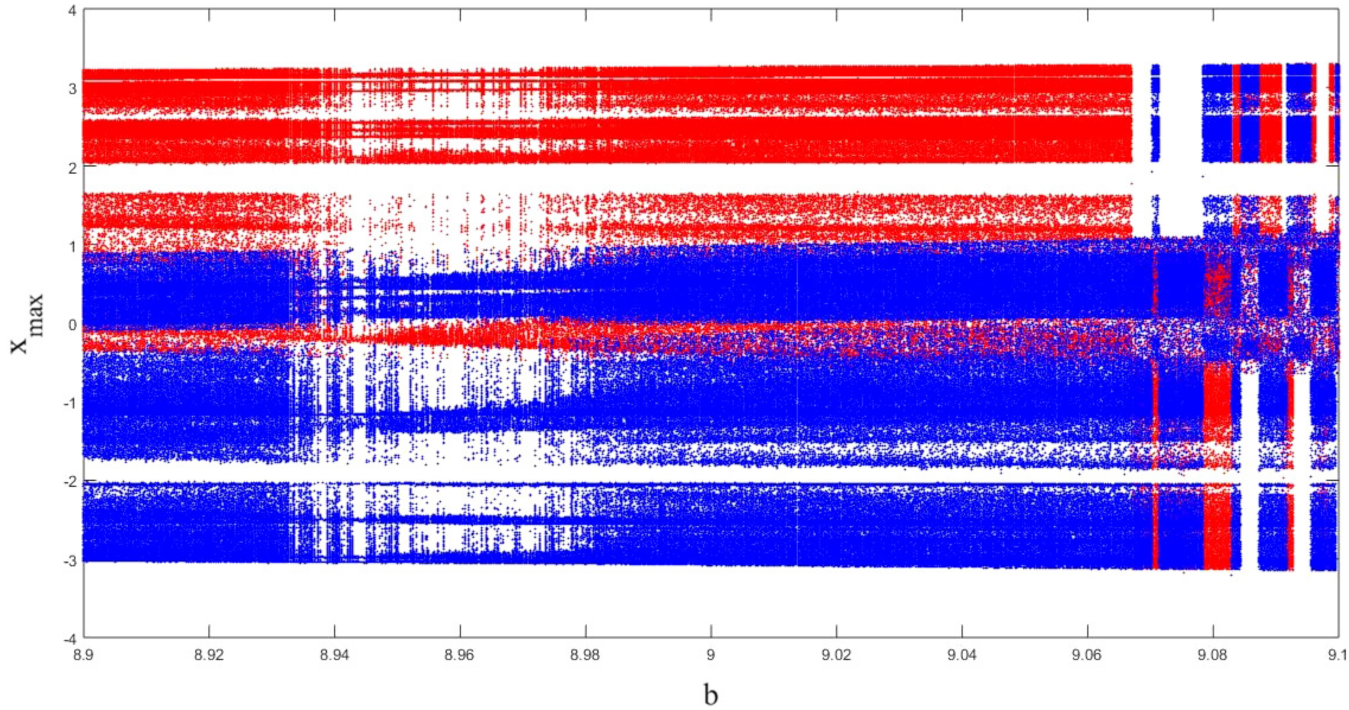


Fig. 9. Multistability in FOCSC system with respect to changing parameter b. The red plot shows a bifurcation diagram with the forward continuation and the blue plot shows a bifurcation diagram with the backward continuation.

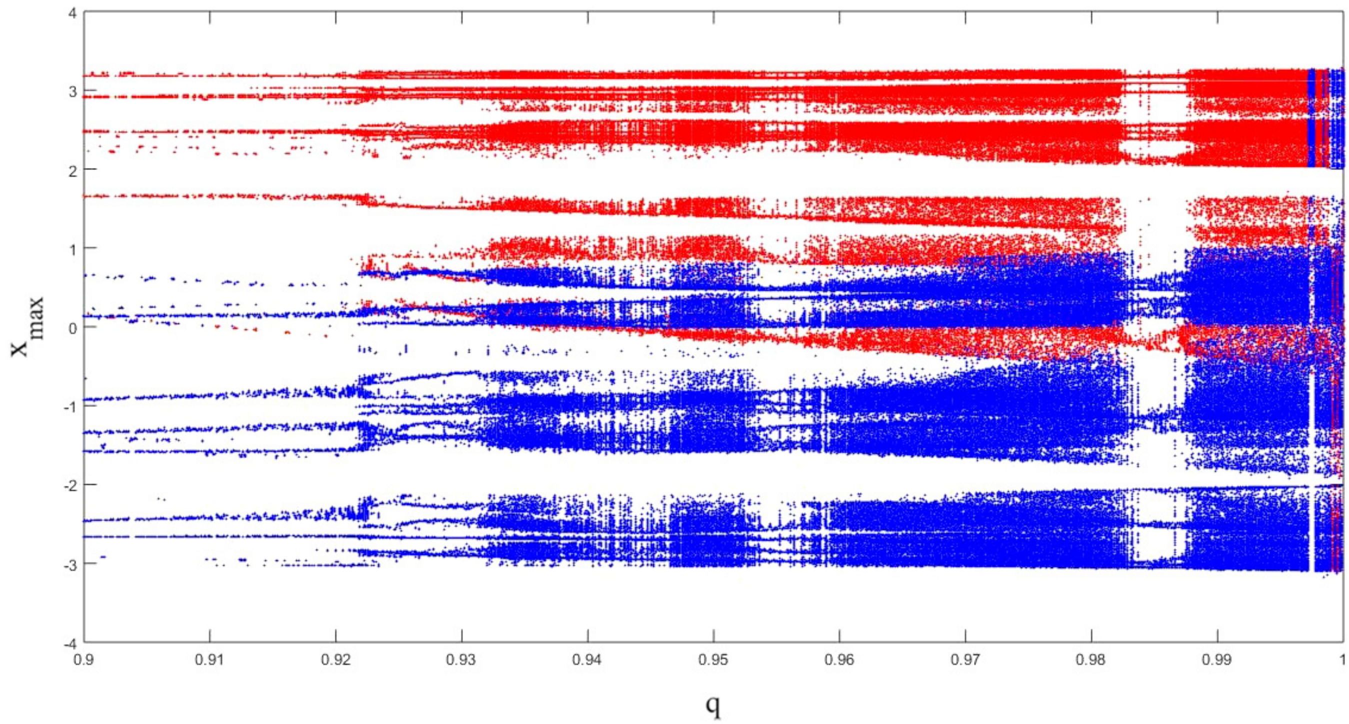


Fig. 10. Multistability in FOCSC system with respect to changing q. The red plot shows a bifurcation diagram with the forward continuation and the blue plot shows a bifurcation diagram with the backward continuation.

3.3. *Bifurcation and multistability*

To investigate the impact of parameters and fractional orders on the FOCSC system, we derive the bifurcation plots as shown in Figs. 7 and 8. As can be seen from them, the FOCSC system shows multiple chaotic regions and crises for changing parameters b and q.

Like multistability in the integer order analysis which has been discussed in Sec. 2, forward and backward continuation methods are used to plot the bifurcation diagrams of the FOCSC system with both parameter b and fractional order q. Figures 9 and 10 show multistability of the system with respect to changing parameters b and q, respectively (the red plot shows forward continuation and the blue plot shows backward continuation). Initial conditions for the first parameter are $[3.5, 2.5, 1.5, 0.5]$ and the final state at each parameter is considered as the initial state for the next parameter. The existence of multistability with various fractional orders is a novel feature which is discussed in FOCSC system. As we know, this feature was not recorded earlier in literature for a fractional order cyclic symmetry system.

4. Circuit Schematic of the Circulant Chaotic System

Circuit implementation of mathematical models is very important in practical applications [Banerjee *et al.*, 2012; Li *et al.*, 2017; Zhang & Liao, 2017; Abdullah *et al.*, 2012; El-Latif & Niu, 2013; Zhang *et al.*, 2014]. Moreover, previous researches have examined the effectiveness of the operational-amplifier approach in implementing chaotic circuits [Akgul *et al.*, 2016b; Akgul *et al.*, 2016a]. Therefore,

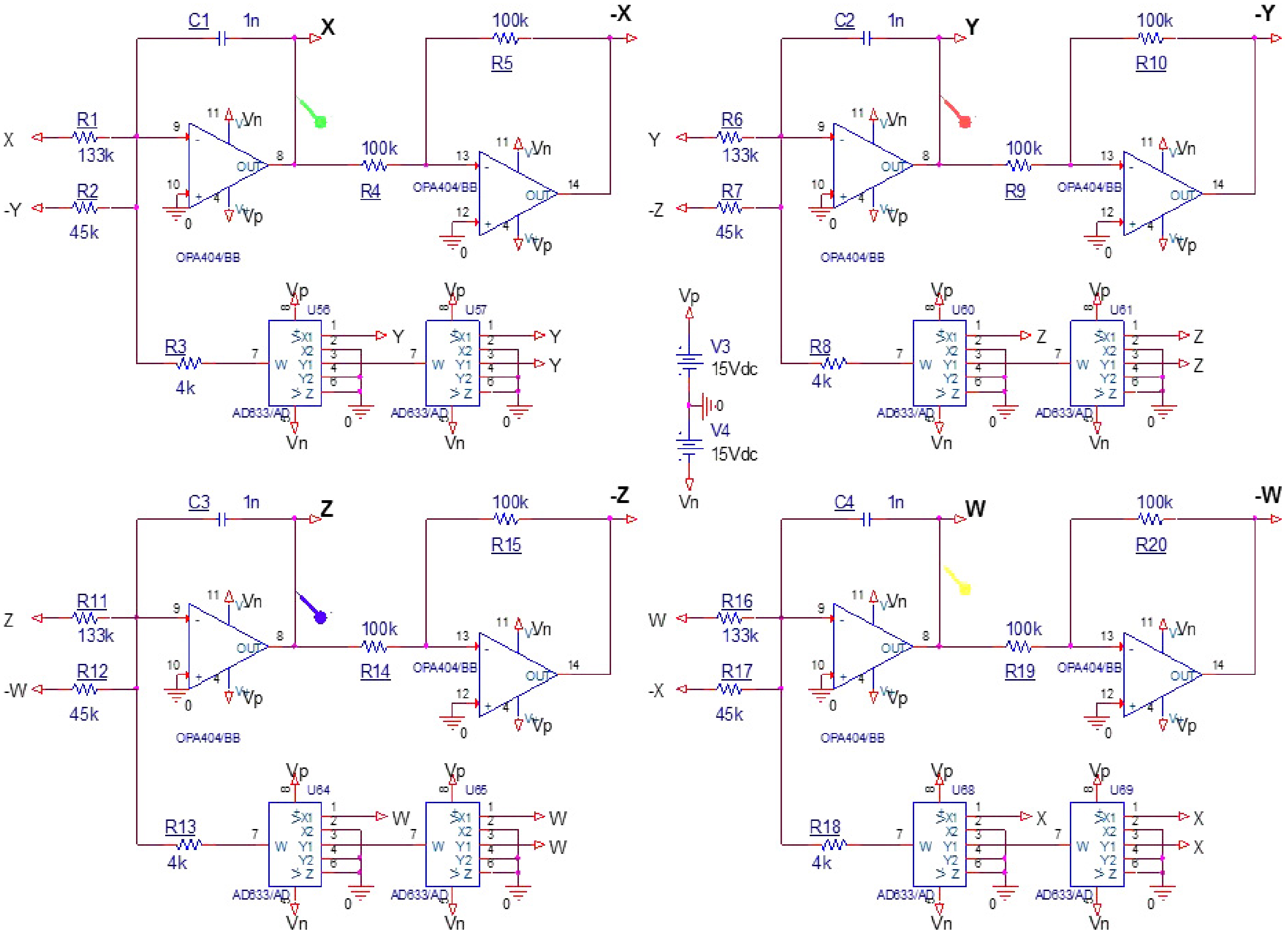


Fig. 11. The circuit schematic of the CSC system based on the operational-amplifier approach. The power supplies are ±15 Vdc.

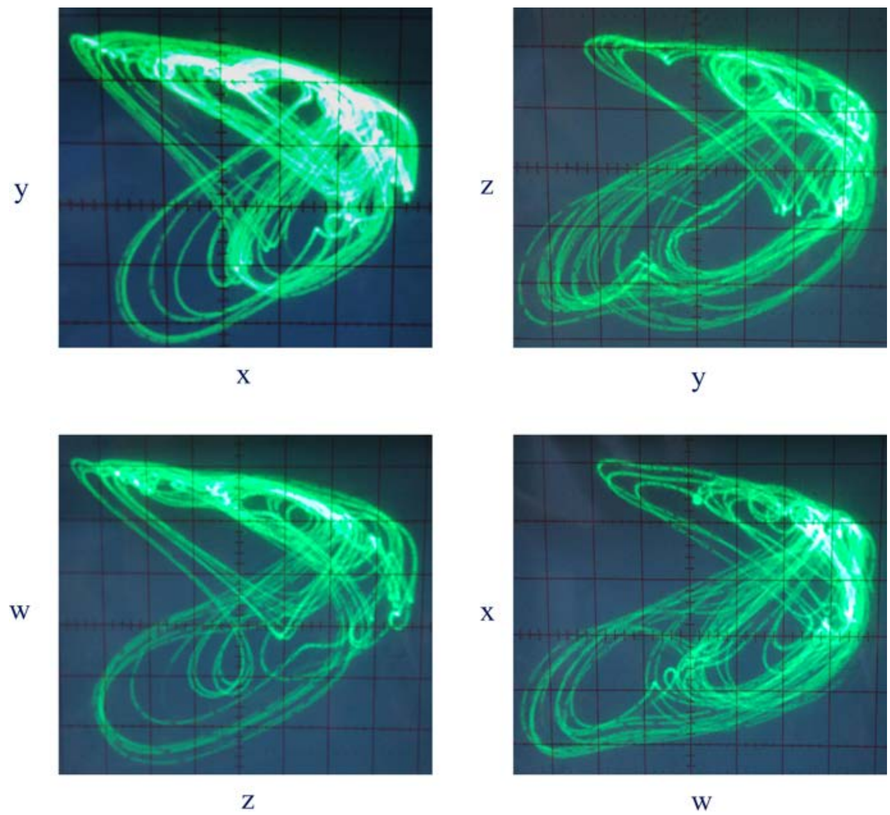


Fig. 12. Experimental phase portraits of the real circuit on the oscilloscope with initial conditions $[0.4, 0.1, 0.1, 0]$.

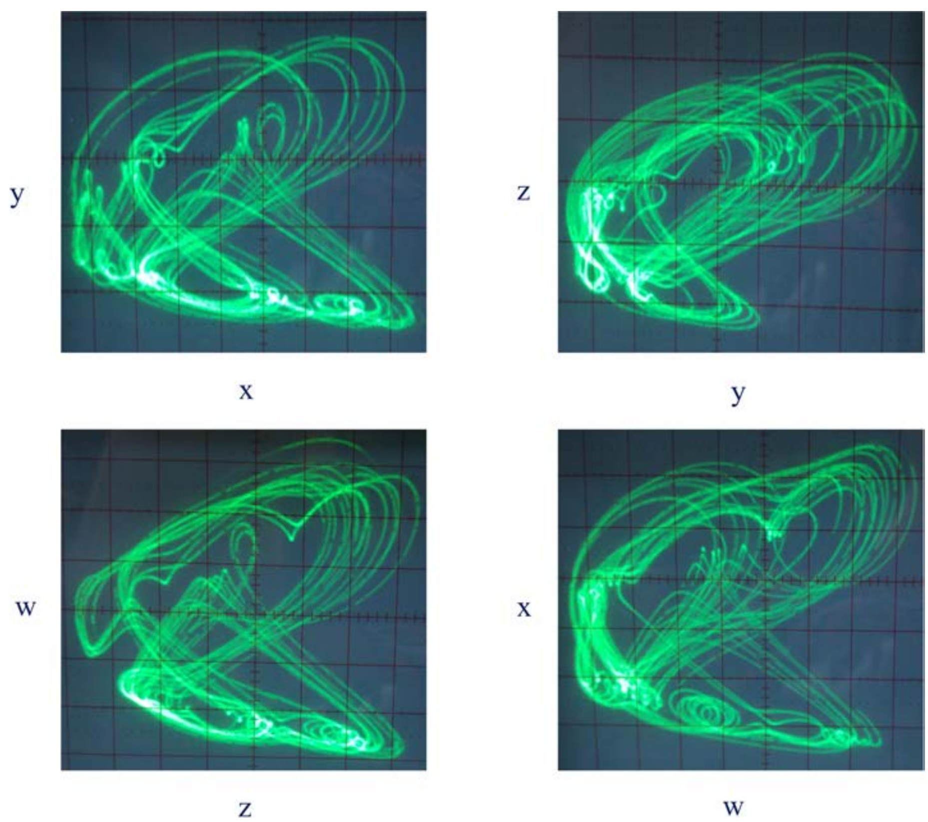


Fig. 13. Experimental phase portraits of the real circuit on the oscilloscope with initial conditions $[-0.4, -0.1, -0.1, 0]$.

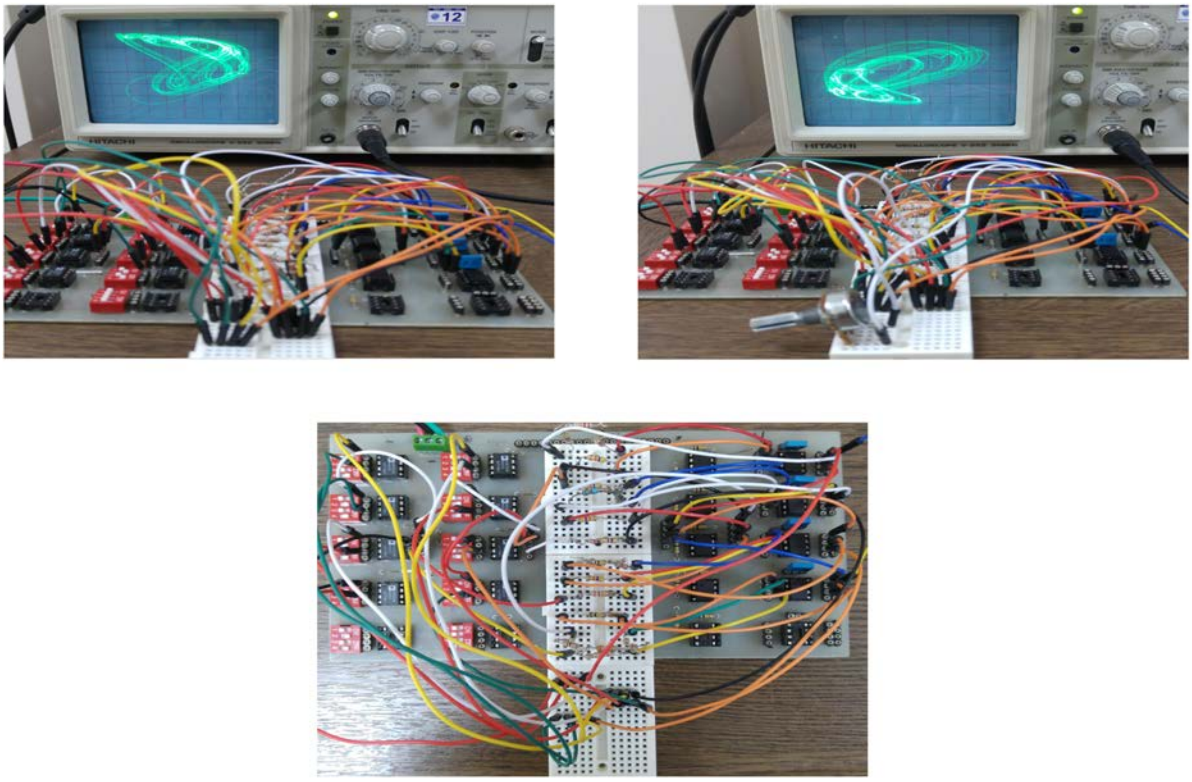


Fig. 14. The real circuit of the CSC system which was implemented on an electronic card.

by using this approach, the circuit which implements the CSC system has been designed as shown in Fig. 11. The values of components are selected as follows: C1 = C2 = C3 = C4 = 1 nF, R1 = R6 = R11 = R16 = 133 kΩ, R2 = R7 = R12 = R17 = 45 kΩ, R3 = R8 = R13 = R18 = 4 kΩ and R4 = R5 = R9 = R10 = R14 = R15 = R19 = R20 = 100 kΩ.

We have implemented the circuit on an electronic card with symmetric initial conditions $[\pm 0.4, \pm 0.1, \pm 0.1, 0]$. Experimental phase portraits of the CSC system on the oscilloscope with symmetric initial conditions are shown in Figs. 12 and 13. In addition, the real circuit on the electronic card is shown in Fig. 14. Experimental results indicate that the circuit displays coexisting chaotic attractors.

5. Conclusion

In this paper, a new 4D circulant chaotic flow has been proposed. The system has presented two symmetric chaotic attractors which coexist. The bifurcation diagram of the system for changing parameter b was discussed. The proposed system has shown multistability in an interval of parameter b. Investigating the bifurcation diagram of the system with two forward and backward continuation methods has shown that the system is multistable in some intervals of parameter b. Also, fractional order model of the proposed system has been investigated in this paper. The fractional order system has shown multistability as well as integer order system. Finally, circuit implementation of the integer order system has been studied, and the circuit implementation of the FOCSC system is left for future work.

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Appendix A

The PECE method for the FOCSC system as described in [Diethelm, 2010].

Let us define a fractional order dynamical system with order q as,

$$D^q x = f(t, x), \quad 0 \le t \le T \tag{A.1}$$

where $x^k(0) = x_0^k$ for $k \in [0, n-1]$. Equation (A.1) is similar to the Volterra integral equation [Diethelm & Ford, 2002] given by,

$$x(t) = \sum_{k=0}^{n-1} x_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau_1, x)}{(t-\tau)^{1-q} d\tau_1}, \tag{A.2}$$

where $h = \frac{T}{N}$, $t_n = nh$, $h \in [0, N]$ and $\tau_1 = \frac{\tau}{h}$.

The discrete form of (A.2) can be defined as,

$$x(n+1) = \sum_{0}^{n-1} x_0^k \frac{t_n^k}{k_1} + \frac{h^q}{\tau(q+z)} f\left(t_{n+1}, x_n'(n+1) + \frac{h^q}{\Gamma(q+z)} \sum a_{j,n+1} f(t_j, x_n(j))\right) \tag{A.3}$$

where

$$a_{j,n+1} = \begin{Bmatrix} n^{q+1} - ((n-q)n+1)^{q+1}, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{Bmatrix},$$

$$x_n'(n+1) = \sum_{k=0}^{n-1} x_0^k \frac{t_n^{k+1}}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_n(j)) \quad \text{and}$$

$$b_{j,n+1} = \frac{h^q}{q}((n-j+1)^q - (n-j)^q).$$

Using the above definitions (A.2) and (A.3), the FOCSC system can be defined as,

$$x(n+1) = x(0) + \frac{h^{q_x}}{\Gamma(q_x+2)} \times \left[(ax'(n+1) + by'(n+1) - y'^3(n+1)) - \sum_{j=0}^{n} \chi_{1,j,n+1}(ax'(n) + by'(n) - y'^3(n))\right]$$

$$y(n+1) = y(0) + \frac{h^{q_y}}{\Gamma(q_y+2)} \times \left[(ay'(n+1) + bz'(n+1) - y'^3(z+1)) + \sum_{j=0}^{n} \chi_{2,j,n+1}(ay'(n) + bz'(n) - z'^3(n))\right]$$

$$z(n+1) = z(0) + \frac{h^{q_z}}{\Gamma(q_z+2)} \times \left[(az'(n+1) + bw'(n+1) - w'^3(z+1)) + \sum_{j=0}^{n} \chi_{3,j,n+1}(az'(n) + bw'(n) - w'^3(n)) \right]$$

$$w(n+1) = w(0) + \frac{h^{q_z}}{\Gamma(q_w+2)} \times \left[(aw'(n+1) + bx'(n+1) - x'^3(z+1)) + \sum_{j=0}^{n} \chi_{4,j,n+1}(aw'(n) + bx'(n) - x'^3(n)) \right] \tag{A.4}$$

where

$$\begin{aligned} x'(n+1) &= x(0) + \frac{1}{\Gamma(q_x+2)} \sum_{j=0}^{n} \theta_{1,j,n+1}[ax(j) + by(j) - y^3(j)] \\ y'(n+1) &= y(0) + \frac{1}{\Gamma(q_y+2)} \sum_{j=0}^{n} \theta_{2,j,n+1}[ay(j) + bz(j) - z^3(j)] \\ z'(n+1) &= z(0) + \frac{1}{\Gamma(q_z+2)} \sum_{j=0}^{n} \theta_{3,j,n+1}[az(j) + bw(j) - w^3(j)] \\ w'(n+1) &= w(0) + \frac{1}{\Gamma(q_w+2)} \sum_{j=0}^{n} \theta_{4,j,n+1}[aw(j) + bx(j) - x^3(j)] \end{aligned} \tag{A.5}$$

$$\chi_{i,j,n+1} = \left\{ \begin{array}{ll} n^{q+1} - (n-q)(n+1)^{q+1}, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{array} \right\}$$

$$\theta_{i,j,n+1} = \frac{h^q}{q}((n-j+1)^q - (n-j)^q), \quad 0 \le j \le n \text{ for } i = 1,2,3,$$

q takes the respective values of q_x, q_y, q_z, q_w depending on the state variables.